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Wittgenstein, Realism, and Mathematics

In previous papers[1], I have argued that Philosophical Investigations and most of Wittgenstein's Remarks on the Foundations of Mathematics do not represent a generally antirealist philosophy of any kind. That doesn't mean that there are no problems with Wittgenstein's remarks, and we shall soon see that at least one of them does flirt with a mathematical form of verificationism. In particular, it does seem that in 1944 (when he wrote the remark that even God's omniscience cannot decide whether people would have reached '777' in the decimal expansion of π "after the end of the world" if they haven't reached it before that[2]) that there was a moment at which Wittgenstein thought that a mathematical proposition cannot be true unless we can decide that it is true on the basis of a proof or calculation of some kind. One might well ask, "How can such a view not spring from antirealism?"

What I shall argue, in brief, is that the mistake in the remark in question [Remarks on the Foundations of Mathematics, V, §34] represents a combination of genuine insight with an inadequate knowledge of actual mathematical practice and especially of sciences which depend on mathematical practice, in particular mathematical physics.[3]

In fact, a most striking fact about the Remarks on the Foundations of Mathematics from the point of view of a mathematical physicist or a mathematician is that in spite of the philosophical importance that Wittgenstein himself attaches to the application of mathematics outside of mathematics, his examples of such application are remarkably trivial. Apart from applications of arithmetic to results of counting, and a few scattered examples of applications of geometry, there are very few examples of applications in the Remarks. In particular, as I shall shortly explain, crucial difficulties for the view that mathematical truth cannot transcend provability can be discovered from an examination of the ways in which mathematics is used in mathematical physics. But apart from a couple of examples from engineering[4], the only example which might even be regarded as an application in mathematical physics in the whole of the Remarks that I was able to find is a trivial application to the orbit of a comet.[5]

In a way, of course, this is not surprising. Wittgenstein was trained as an engineer; and, indeed, one gets the feeling in all of his writing on the philosophy of mathematics that he imagines that the application of mathematics to empirical material either consists in reading off geometrical relations from pictures or comparing the results of calculations with the results of counting and measurement. These are exactly the ways in which an engineer applies mathematics. I don't mean to suggest that Wittgenstein did not know that there are much more complicated uses of mathematics than these in mathematical physics. What I think likely is that he had no idea of the detailed nature of those applications, and that he assumed that while they might, indeed, be more

complicated than the trivial ones that he used as examples, that is, indeed, all they are -- more "complicated"; that is to say that nothing of philosophical interest could be lost by confining attention to the few and trivial applications that he does discuss. If so, I believe that he made an error; but an error that didn't flow from a metaphysical position.

I want to emphasize, however, that Wittgenstein's remarks about mathematics involve numerous insights. I shall begin by describing just two of these insights.

Two Wittgensteinian insights

(1) Mathematical propositions wouldn't be propositions [Sätze] -- i.e. meaningful statements -- if mathematics were not applied outside of mathematics. (-- Note that this does not mean that every single mathematical proposition need have such applications to be meaningful.)

If mathematics were not applied outside of mathematics if there weren't "mixed" statements – empirical statements which contain mathematical terms - then there would be no reason to view it as more than a game (one in which, for example, we are allowed to write certain marks on paper when the marks are "axioms" or when other players have written certain other marks on paper -- or in a "solitaire" version of the game, when we ourselves have written certain other marks on paper). The question of the "truth" of anything that is produced in the course of the game would be as silly as the question of whether a move in a chess game is "true". –Insight (1) is, of course, one that Wittgenstein has in common with Frege and Russell, who stressed the importance of the fact that numbers can be used to count things, and for that reason number words occur constantly in empirical statements.[6]

The fact that Wittgenstein shares this insight with the great logicists is the reason that it is a mistake to claim (as Michael Dummett does) that Wittgenstein believed that understanding mathematical propositions is just a matter of understanding the proof procedures by which we verify them. Indeed, at one point Wittgenstein suggests that a mathematical "proposition" might have a proof but no real meaning, precisely because we would have no idea of how to apply it.[7]

(2) Commonsense realism with respect to “rule following” -- which, I argued in a previous paper[8], Wittgenstein defends -- does not, in and of itself, commit us to views about infinity (i.e., about the sort of problem that motivates Kripke's discussion.[9])

Kripke’s celebrated "skeptical problem"[10] is initially formulated as the question, how can our understanding of a rule determine what is true in infinitely many cases -- and so in more cases than it is physically possible for human beings to "get to". But Wittgenstein is quite clear that when someone (say a child) learns to follow the rule "add two", he learns to follow a practice which it is possible for human beings to engage in; but such a practice is not one which extends to infinitely many cases, because human beings cannot -- not really -- add two to infinitely many numbers.

Consider, for example, what Wittgenstein says about a child's understanding of infinity (Remarks, Part IV, §14):

> "Suppose children are taught that the earth is an infinite flat surface; or that God created an infinite number of stars; or that a star keeps on moving uniformly in a straight line, without ever stopping.
>
> Queer, when one takes something of this kind as a matter of course, as it were in one's stride, it loses its whole paradoxical aspect. It is as if I were to be told: don't worry, this

> series, or movement, goes on without ever stopping. We are as it were excused the labor of thinking of an end.
>
> 'We won't bother about an end.'
>
> It might also be said: 'for us the series is infinite'.
>
> 'We won't worry about an end to this series; for us it is always beyond our ken.'"

It is significant that Wittgenstein says "Suppose children are taught"; that is to say, this last quotation is not a comment on how a sophisticated mathematician might understand these statements, but about how a child might understand the statement that a series (e.g., the series 1,2,3......) is infinite, or that it "has no end".[11]

The point Wittgenstein makes here can be put this way: that when we speak of a human being as being able to follow a rule, in the ordinary applications of that concept we are not required to ascribe to the person (in this case, the child) the mathematical notion of infinity, or the notion that the rule determines what it is correct to say in cases that it is not actually possible to get to, etc..

Kripke, on the other hand, runs together two different questions: the question how our understanding of a rule determines what we count as correct in actual human practice, and the question of what the mathematical consequences of the rule are. In posing his "skeptical problem", Kripke takes it as evident that an arbitrary sum is correct if and only if the answer is determined by the rule; but this is a way of thinking that Wittgenstein criticizes from the very begin of the Remarks. As Wittgenstein makes clear[12], to criticize this is not to say that one cannot introduce a mathematical sense of "determine" according to which, if we take the "rule" of addition to be, e.g., the definition of addition by primitive recursion, we can say that "every correct sum is determined by the rule". But to

say this is to make a mathematical comment about addition; it is not to speak to any of the philosophical problems about mathematics, because just the notion that puzzled us, the notion of mathematical correctness, has been simply taken for granted.

Thus, we might respond to Kripke as follows: "Professor Kripke, we propose to use the expression A is determined as the correct answer by the rule R only when it is possible for human beings to actually calculate A by using the rule R; and it all other cases we will use the different expression A is a mathematical consequence of the rule R. Now, one of the problems you raised, namely how does a rule that we grasp manage to determine what is correct in infinitely many cases, simply does not arise; for if 'determine' means what we have just proposed it ought to mean, then a rule does not determine what it is correct to say in infinitely many cases (although it is "always beyond our ken" just where we will cease to be able to actually apply it, and hence where it ceases to determine an answer). And if you reformulate your puzzle, by saying "How can a finite rule have infinitely many different mathematical consequences?', then that seems to be a mathematical question, and one whose answer is trivial." I do not mean to say, of course, that this response completely “defuses” Kripke’s worries -- those worries have complex sources -- but at least one strain in Kripke’s complex argument does seem to depend on the conflation of these two senses of “determine”.

A further remark in this connection: if one accepts what I described as Wittgenstein's defense of our commonsense realism about rule following, then one could be led to suppose that the notion of being provable is an unproblematical notion, because to be provable in a formal system, we might say, is just to be a sentence which is obtainable as the last line of a longer or shorter proof (or as the bottom of a larger or smaller proof tree) by following

certain rules. But we can now see that that is too simple a response. In particular, we can object that Wittgenstein's commonsense realism about rule following cannot give us the mathematical notion of "provability" (if we take the notion of “following a rule” as referring to something that it is actually possible for humans to do, just as we took the notion of being “determined by a rule” to refer to something that it is actually possible for human beings to find out); at best it can give us the ordinary empirical notion of "provability". (Of course in daily life we sometimes use the word "provable" in one way – corresponding to the notion of being “determined by the rules”, in the terminology just introduced - and sometimes in the other – corresponding to the notion of being a mathematical consequence of the rules - which adds to the confusion.) When I said earlier that Wittgenstein may have thought that a mathematical proposition cannot be true unless we can decide that it is true on the basis of a proof or calculation of some kind, I meant the modal word “can” to refer to what is "humanly" possible; for to say that that a mathematical proposition cannot be true unless it "decidable" in the mathematical sense -- the sense in which it might be decidable even though the shortest proof or disproof was longer than the number of elementary particles in the universe -- would be open to the objection that I just made against Saul Kripke, the objection that to understand that notion of decidability (or of provability or of disprovability) requires being able to understand the sort of mathematical notion Wittgenstein wishes to investigate.

Dummett’s interpretation of Wittgenstein on truth and provability in mathematics

Michael Dummett[13] has claimed that Wittgenstein held the view that being actually proved is a necessary condition for mathematical truth. But even though here and there in Wittgenstein's Nachlass one can turn up a note that shows that

Wittgenstein "played with the idea" at certain moments in his life, I do not find anything in the Remarks which should be construed as committing Wittgenstein to such a radical view. For example, when we read the passage written in 1944 (Remarks, Part V, §34) that I mentioned at the beginning of this lecture, let us pay close attention to what Wittgenstein says:

> Suppose that people go on and on calculating the expansion of π. So God, who knows everything, knows whether they will have reached '777' by the end of the world. But can his omniscience decide whether they would have reached it after the end of the world? It cannot. I want to say: Even God can determine something mathematical only by mathematics. Even for him the rule of expansion cannot decide anything that it does not decide for us.[14]

Of course, human beings cannot possibly calculate at all "after the end of the world" -- especially if, as is reasonable to suppose, the "end of the world" means the end of space and time.[15] Thus the most reasonable interpretation is that Wittgenstein meant us to suppose that human beings go on calculating to the very limit of what is possible for such beings. That is, Wittgenstein is not here -- contrary to Dummett's interpretation -- saying that not even God could know what human beings would count as a correct calculation prior to their actually accepting it as correct (which would involve attributing a strongly "antirealist" attitude towards counterfactuals to Wittgenstein, on no evidence that I can see); he is saying -- as he goes on to make clear -- that not even God can decide whether the pattern does or does not occur in the expansion of π except by a calculation which is actually -- not just "mathematically" -- possible.

But even if Dummett’s interpretation is wrong, it still remains to say how, given what I have been calling his "commonsense realism about rule following", a commonsense realism which, we have just seen, does not in and of itself involve

one in the sorts of problems that Kripke raises, because it does not require a notion of infinity beyond the notion Wittgenstein says a child might have ["for us the end is always out of reach"], and given a robust insistence on the fact that mathematical propositions are statements with sense only because mathematical concepts have applications in the realm of the <u>nonmathematical</u> -- given that these are both genuine insights <u>and</u> genuinely commonsensical, how Wittgenstein can have been led to flirt with the view that provability (in the ordinary empirical sense) is a necessary condition for mathematical truth.

My answer is that if one makes the mistake of supposing that the sorts of examples of the applications of mathematics outside of mathematics that Wittgenstein uses <u>exhaust</u> the philosophically relevant sorts of examples there are, then the position I have ascribed to Wittgenstein can appear quite attractive; for there is nothing in those sorts of applications that would <u>require</u> us to suppose that humanly[16] unprovable mathematical propositions can have a truth-value. The notion that a humanly unprovable mathematical proposition can have a truth-value. That notion can then appear (as I believe it did appear to Wittgenstein when he wrote this paragraph) to be just as a piece of metaphysical fabulation. The situation is quite different, however, when we consider a very different sorts of example, one taken from serious mathematical physics.

Where Wittgenstein went wrong

When we make the statement that a physical system obeys a certain equation (this is an example of what I earlier called a “mixed” statement, an empirical statement that contains mathematical terms), for example, when we say that the state-vector of a physical system obeys the Dirac equation, or, in Newtonian physics, when we say that gravitational forces obey the Newtonian Law of Gravitation, or even when we say that a certain phenomenon obeys the

wave equation, what is the situation? As long as we accept the correctness of Newton's Law of Gravity, we are committed to the statement that the evolution of an N-body system will be in accordance with the solutions to the appropriate system of differential equations. In general, any application of mathematical physics to a physical system involves treating the system as behaving in accordance with some equations – usually differential equations – or others. If a physicist believes the equations are correct, and furthermore that they describe the behavior of the system at each time-point, then she is committed to the claim that those equations have solutions (in real numbers, or, in certain cases, in complex numbers) for each real value of the time parameter t.

Now, suppose that the solution to the equations, in a particular physically given case, for a particular rational value of t is in a certain rational interval, say, between r_1 and r_2, where r_1 and r_2 are rational numbers. That is suppose the statement[17] "the solution to such and such equations for the given value of t is in the interval between r_1 and r_2" is true. If truth is the same as provability (in the "ordinary empirical sense" of provability), then what the physicist is committed to, if she believes the view that Wittgenstein flirted with in RFM, V, §34, is that it is possible to calculate the solution to the equations in question for the specified value of t, and that the solution will be found to lie in the interval between r_1 and r_2. But, given present knowledge, this is something a sensible physicist better not commit herself to!

Since this point is at the heart of my argument, let me state it again for clarity: if truth is coextensive (in the mathematical case) with empirical possibility of being proved by a calculation or a proof from axioms, then no mathematical statement can be true but not demonstrable by calculation or proof. Not even "God's omniscience" can know the truth value of such a humanly undecidable

statement, which is to say it doesn't have a truth value. If the statement is that the equations of motion of a system **S** have a solution (say "P(t) is in the interval between 3.2598 and 3.2599") where P is some physical parameter, then if it is not physically possible for human beings to compute P(t), or to prove by some deduction from acceptable axioms that P(t) is or isn't in the interval between 3.2598 and 3.2599, then there is no fact of the matter as to whether P(t) is in that interval or not. But to accept this is precisely to be a verificationist in one's physics. It is to give up a claim which is part of our best physical theory of the world, the claim that the equations of that theory describe the behavior of certain systems accurately and completely, in the sense that those equations have solutions for each real value of the time parameter t, and those solutions give the value of the physical parameter P in question even if it isn't feasible to verify that they do in certain cases. Systems of equations are, on a verificationist view, just prediction devices, and when it is not feasible to derive a prediction from them (even if we are allowed to go on calculating until "the end of the world"), then there is nothing that they say about the case in question.

But why do I say that the physicist "better not commit herself to" the claim that it is physically possible to calculate the solution to the differential equations of physics for arbitrary specified values of t? One important reason has to do with chaos (a phenomenon represented mathematically by certain kinds of differential equations): when a phenomenon is sufficiently "chaotic", it may be "empirically" (though not "mathematically") impossible to actually calculate the time evolution of certain parameters even though we know the equations they obey. An additional (possible) reason has to do with nonrecursiveness: it is to this day quite unknown whether the solutions to, for example, the Newtonian gravitational equations are recursively calculable even when N=3. [18] In addition, it

is known that there are cases in which it has been proved that the solutions to the wave equation are not recursively calculable, even given recursive initial data.[19] Indeed, we do not even know whether the values of physical magnitudes at specified future times are, in general, effectively calculable to even one decimal places when those magnitude obey these equations.

In sum, short of being a verificationist about physics, one cannot consistently sustain the identification of mathematical correctness with provability and with calculation that seems to be asserted in V, §34 of RFM.

How might what Wittgenstein wrote in RFM, V, §34 be defended against this criticism?

As far as I can see, there are only three ways in which a defense of V, §34 of RFM that one could take seriously might go. First, one might look for a different interpretation – one according to which when Wittgenstein asked "But can his [God's] omniscience decide whether they would have reached it after the end of the world?" and answered "It cannot" he was not asserting that there is no fact of the matter as to whether 777 occurs in the decimal expansion of π unless human beings are able to show that it occurs "before the end of the world" (or, presumably, to show that it doesn't).[20] I anticipate that some of you at this conference will suggest such possible interpretations. But I myself find the fact that Wittgenstein says not just that God could not decide this, but that his omniscience could not, makes it almost crystal clear that V,§34 means that the statement that 777 occurs in the expansion is neither true nor false in the envisaged circumstance.

Assuming, for the sake of the argument anyway, that my interpretation of V, §34 is the right one, the other two ways of defending it against my criticism that I can envisage are the following:

(1) One can point out (correctly) that the physical theories we have today and are likely to have in the future are <u>idealizations</u> and not literally correct descriptions of the physical universe. One might argue that this means the entire problem I have raised does not really arise.

(2) One can "bite the bullet" and argue that one should be a "verificationist" in physics, at least to the extent of agreeing that what physical theories say about physical reality does not go beyond what it is (physically) possible for human beings to calculate on the basis of those theories.

In the next two sections I shall consider these two defenses in turn.

Physics is idealized. Does that make a difference to the argument?

It is quite true that our most fundamental physical theories – quantum mechanics and General Relativity – cannot be regarded as perfectly correct as they stand. For one thing, neither satisfactorily incorporates the other. ("String theory" might alter this situation, if some version of it succeeded. However, the "measurement problem" of quantum mechanics would still be unsolved – in my opinion, at least – and it may well require a deeper theory than any we have today to resolve it.) For this reason, we can regard these "fundamental theories" only as idealizations, or approximately correct descriptions of the evolution of physical systems in time. Does this undermine the argument for interpreting the equations of those theories realistically?

I cannot see that it does. In a host of ordinary situations (situations in which the curvature of space time plays a part only in a way which is well-modeled by existing theories of "quantum gravity", such as the familiar terrestial situations in which classical mechanics works quite well) the correctness of

quantum mechanics has been confirmed to many decimal places. On the other hand, the phenomena I pointed to above – phenomena such as “chaos” and the possible non-recursivities in the solutions to certain differential equations – prevent us, in many cases from calculating the values of certain parameters to even the first decimal place! Thus even if we replace the claim that the wave equation or the gravitational equation or the equations which govern a complicated “chaotic” phenomenon are precisely correct (a claim which is certainly not true) with the claim that (in such and such a situtation) the equations yield the right answer to, say, four or five decimal places (a claim we believe to be true), the same problem arises. On the one hand, our best theory of the world includes the claim that these equations do have solutions, and those solutions yield the values of the parameters, whether it is feasible to calculate those solutions or not. On the other hand, RFM, V, §34, as I am interpreting it, says that if it not feasible to compute a solution, even if we are allowed to go on computing to the “end of the world”, then there isn’t one. The statement that any sufficiently small rational interval contains the solution is neither true nor false, if it is not feasible to determine that it does, on the view expressed by this paragraph. But then the equations of physics are not what we take them to be – a nomological account of how physical systems evolve in time. It is not, for example, (if RFM, V, §34 is right) that it is too hard (because of chaos, etc.) to determine how a complicated system evolves: the equations don’t predict how it evolves. What they predict depends on our human powers of computation.

If this had been Wittgenstein’s considered view, as opposed to an isolated remark, then his claim that he offered no theses” in philosophy would have been deceptive. (But the claim about what omniscience can’t decide in RFM, V, §34 is, fortunately, one that Wittgenstein does not repeat when touches on the same question in Philosophical Investigations §516.)

Should we "bite the bullet"?

Should we, however, say that one should be a "verificationist" in physics, at least to the extent of agreeing that what physical theories say about physical reality is exhaustively determined by what it is (physically) possible to calculate on the basis of those theories? Can one coherently be this sort of verificationist without being a verifications tout court?

Ian Hacking, for example, has proposed that we can and should be "realists" about the existence of positrons but not about the theories about them.[21] In brief, Hacking claims that being a realist about the entities physical theories postulate doesn't have to mean being a realist about the theories. But this attempt to disjoin realism about physical entities and realism about theories does not work – it fails for a very Wittgensteinian reason. The notion of a "positron" (positrons were Hacking's example[22]) depends on the language game to which positron talk belongs, and it is the quantum theory that structures that very strange game. We speak of "spraying" positrons in a certain experiment Hacking describes, for example. But we may also (depending on the experiment) speak of "spraying a superposition of three positrons and 5 positrons" – which does not mean "we sprayed three positrons and we sprayed five positrons on top of them" (that would be spraying eight positrons, something which the statement that we sprayed a superposition of three positrons and five positrons rules out), and which does not mean "we sprayed three positrons or we sprayed five positrons", and which does not mean anything else that a classical physicist could understand. Positron talk has an entirely novel grammar, and that grammar is provided by the theory. When Hacking says that he is "a realist about positrons", he doubtless means that he believes that "positrons exist". But that statement means literally nothing apart from a specification of what concepts do and what concepts do not apply to positrons. If, for example, Hacking thinks that

positrons exist in the sense of being countable entities with a position in space at each time, then he is not an antirealist about quantum theory at all: he simply thinks quantum theory is false, and that positrons are classical particles. But I am sure that isn't what he thinks! -The fact is that to say that one is a realist about positrons but not about quantum theory or any other physical theory would be to say nothing. One can no more take the existence of positrons seriously without taking the conceptual apparatus that goes with positron talk seriously than one can take the notion of baldness seriously while rejecting the concept of hair.

In sum, being an verificationist about the theory precludes any substantial sense of being a realist about the entities. This can be seen from the sort of example I used to put pressure on Wittgenstein's remark RFM, V, §34. Suppose that the temperature of the gas in a certain small region in the sun cannot be measured directly or indirectly.[23] To say that in that case the notion of the temperature of the gas in the region is meaningless would be classical verificationism. Now, suppose that although the temperature of the gas cannot be measured directly, there is an equation whose solution is the temperature in question. Saying that whether the notion of the temperature of the gas in the region is meaningful or not depends on whether a certain calculation (which is mathematically well defined, we will suppose) is feasible or not would be just as verificationist would it not?

Alternatively, suppose we say (assuming that there is nothing relevantly wrong with our theory) that the notion of the temperature of the gas in the region X is a perfectly meaningful one, and that, as common sense realists with respect to temperature, we of course believe that there is a fact of the matter as to whether the temperature of X is between A and B or between C and D (where A.B,C, and D are possible temperatures for such a region). But why would we think that the statement that, say, the temperature is between A and B (A < t(X) <

B) has a truth value even if people couldn't find it out, if we reject the idea that the purely mathematical statement "777 occurs in the decimal expansion of π" has a truth value even if people couldn't find it? Both statements, after all, employ mathematical notions. Such a doesn't seem coherent.

Well, why shouldn't one "bite" the bullet and be a verificationist all the way? Enough people have argued that this is what Wittgenstein was, after all. I will not repeat the arguments against verificationism, which are by now familiar to everyone.[24] But I will say a word about my reasons for not reading Wittgenstein as a verificationist[25]

On not reading Wittgenstein as a verificationist

A strong reason for not reading the later philosophy of Wittgenstein as a form of verificationism has already been mentioned in passing: Wittgenstein is insistent that his aim is not to defend any "theses" in philosophy, but simply to teach us to expose disguised nonsense. It is hard to see how any of the different forms of verificationism can be regarded as anything other than a philosophical thesis. Moreover, the philosophers who more than any other have defended a verificationist interpretation of Wittgenstein – Michael Dummett and Crispin Wright – are avowed critics of what Wright calls Wittgenstein's "quietism" and what Dummett has referred to in conversation as Wittgenstein's "opposition to theory". The fact that these verificationist philosophers, brilliant as they are, are so out of sympathy with what was so obviously at the center of the later Wittgenstein's philosophical stance should lead us to view with suspicion their attempts to foist a "theory" on such works as Philosophical Investigations and the Remarks on the Foundations of Mathematics.

Doubtless one reason that Dummett and Wright wish to read verificationism into the later philosophy is that they believe that their own

(different) versions of verificationism are right, and so, in accordance with “the Principle of Charity”, they naturally try to find passages in Wittgenstein which can be read to accord with those versions, and dismiss passages which disclaim the ambition to propound philosophical theses as places where Wittgenstein was “weak”, or confused about the nature of his own best contribution. But for those of us who, like myself, think that verificationism is wrong, the Principle of Charity works the other way: we think that if Wittgenstein doesn’t have to be read as a (crypto-)verificationist, then he shouldn’t be. And, in fact, almost all of the famous passages that were at one time read in a verificationist way (or a “skeptical” way, in Kripke’s case) have been given what seem to me much better readings of a very different sort. I myself have interpreted the “rule-following discusson” in Philosophical Investigations in way which makes clear that if the discussion of rule-following expresses skepticism at all, the "skepticism" is directed at philosophical accounts of rule following, and not at rule following itself. What readers like Kripke have done, I claimed, is to take Wittgenstein to oppose not only metaphysical realism about rule following but also our commonsense realism about rule following, when what Wittgenstein actually doubts is the need for and the possibility of an account of rule following over and above the commonsense account.[26] Stanley Cavell and others have long demolished the idea that Wittgenstein endorsed some version of “behaviourism” in Philosophical Investigations.[27] More and more, it seems that it is only passages in Remarks on the Foundations of Mathematics which are being used to support the ascription of verificationism to Wittgenstein.

Yet here too, at least on a second reading, there is precious little that demands a verificationist reading. The comparison of certain mathematical statements to conventions, for example, is meant as an analogy which highlights certain ways those statements are used, not an explanation of their truth, much

less an attempt to explain “the nature of mathematical truth”, as Yemima Ben Menahem has shown.[28] The “notorious paragraph” on the Gödel theorem[29], which for a long time seemed to me conclusive evidence that Wittgenstein identified mathematical truth with proof admits of a much more interesting (and better) interpretation, as Juliet Floyd and I have discovered.[30] Indeed, Remarks on the Foundations of Mathematics, Part V, §34 seems the only place where, at least for a moment, Wittgenstein expresses what I have called a “verificationist” attitude (towards a possible[31] mathematical truth, not physics, however). Should we conclude, on such slight evidence, that this was Wittgenstein’s considered opinion?

It seems to me that we should not. To see how otherwise to account for RFM, V §34 we need, I think, to do two things: we need to look at the paragraph in question again more carefully, and we need to look carefully at its more cautious twin, §516 in Philosophical Investigations. Here are the two paragraphs in question:

> [RFM, V. §34:] Suppose that people go on and on calculating the expansion of π. So God, who knows everything, knows whether they will have reached '777' by the end of the world. But can his omniscience decide whether they would have reached it after the end of the world? It cannot. I want to say: Even God can determine something mathematical only by mathematics. Even for him the rule of expansion cannot decide anything that it does not decide for us.[32]

> [PI §516:] It seems clear that we understand the meaning of the question: 'Does the sequence 7777 occur in the development of π?' It is an English sentence; it can be shown what it means for 415 to occur in the development of π; and similar things. Well. our

understanding of that question reaches just so far, one may say, as such explanations reach.

What I take to be Wittgenstein's considered opinion is that the very notion of a nonmathematical explanation of the correctness of mathematics is disguised nonsense. As I suggest we read it, what PI §516 tells us is that, just as our explanations of rule following described in PI §208 ("I shall, for instance, get him to continue the ornamental pattern uniformly, when told to do so") suffice perfectly, and there is no "deeper" explanation for philosophy, or future neurology (PI§158), or anything else to give, so our ordinary (mathematical) explanations of mathematical correctness suffice perfectly, and here too there is no "deeper" explanation to be given.[33] A brief look at one well-known attempt at such a "deeper" explanation – David Lewis's – may clarify what I mean.

David Lewis tries to explain why mathematics "works" by postulating a large cardinal's worth of invisible "stuff".[34] But in doing so, what he is embroiled in is a search for non-mathematical reasons that the truths of mathematics are truths. But such a search makes no sense. Showing that it doesn't – in detail – which I don't pretend to do here – would be one way of carrying out the philosophical task that Wittgenstein describes at PI §119 when he writes "the results of philosophy are the uncovering of one or another piece of plain nonsense". But the following is how I would start. I would ask the question: if the vast amount of invisible stuff that David Lewis postulates as part of his explanation of the truth of set theory ceased to exist, would mathematics stop working? If Lewis answers "yes", he will be required to say that we causally

interact with the invisible "stuff" – which he emphatically denies. (And if he had been willing to say that, he would have turned mathematics into a fictitious descriptive science – which is to say, it would have ceased to be mathematics!) But if mathematics would work just as well even if Lewis's "stuff" didn't exist, what explanatory work does the "stuff" do? But I digress.

It seems to me that the reading just proposed perfectly fits PI §516. But what of RFM, V, §34? Part of this paragraph fits the same reading: "Even God can determine something mathematical only by mathematics" fits well with the idea that there is no "foundation" for mathematics, and no explanation of mathematical concepts (including the concept of mathematical correctness) other than the mathematical ones. Where Wittgenstein went astray, I have been suggesting, is in the identification of mathematical correctness with decidability. But is it really an identification?

It is only an identification if we suppose that Wittgenstein thought that the thought-experiment proposed in RFM, V, §34 is really an intelligible one; but the minute we stand back and reflect, this looks quite unlikely, I believe. Is it at all clear what it would mean for human beings to compute until "the end of the world"? Is it at all clear what they would and what they would not succeed in calculating? To suppose that Wittgenstein is, say, defining mathematical truth, e.g., thus:

(I) S [a sentence of mathematics] is true if and only if human beings would, were they to go on and on calculating and writing down proofs, succeed in proving S before the end of the world.

-seems absurd! Thus the assumption that I have proceeded on throughout this lecture, that Wittgenstein thought that mathematical truth is coextensive with something I called "empirical" possibility of (human) proof is enormously implausible. It is time to revise it.

What I suggest instead is the following: Wittgenstein at the moment he wrote RFM, V, §34 meant to say that the idea that there is such a thing as a mathematical truth that utterly escapes the possibility of proof is metaphysical nonsense. I think he was wrong about this for the reasons given above. But to (mistakenly) reject the idea that mathematical truth transcends provability does not require one to think that the notion of "possibility of proof" is itself a completely clear one. (Mathematical truth may also not be a completely clear notion.) The point was not, if I am right, to take a supposedly quite clear notion of "possibility of proof" (in the paragraph, Wittgenstein only considers a constructive proof, via a calculation), and use it to explain mathematical truth; the point was rather to express discomfort with the idea that mathematical truth can completely transcend possible recognition by "us" . I have been arguing that the discomfort should be resisted: we have excellent reasons for thinking that mathematical truth can transcend recognition by "us". But if Wittgenstein succumbed momentarily to a discomfort that is it is extremely easy to feel, that does not mean that he fell into the trap of developing a verificationist philosophy. For, as already mentioned, he simply excised the suspicious part of the remark when he came to write Philosophical Investigations §516. In short, it looks as if he later felt

at least some discomfort with the discomfort he felt when he wrote Remarks on the Foundations of Mathematics, V, §34.

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[1] "On Wittgenstein's Philosophy of Mathematics", The Aristotelian Society, Supplementary volume LXX (1996), 243-264. A much longer version of this paper titled "Was Wittgenstein Really and Antirealist About Mathematics," will appear in McCarthy and Stidd (eds.), Wittgenstein in America (forthcoming).
[2] Remarks on the Foundations of Mathematics, Part V, §34. I quote the paragraphs in question in full later in this essay.
[3] One reason – by no means the only one – that I say that Wittgenstein's examples indicate an inadequate knowledge of mathematical practice is the assumption throughout that what I called "quasi-empirical methods" in [complete ref] play no role. The fact is that even before computers succeeded in showing that "777" and "7777" occur in the decimal expansion of π, every first class number theorist was sure that it did on "quasi-empirical" grounds.
[4] Examples from engineering in Remarks: Part III, §49, Part V, §51.
[5] Part IV, §23, refers to the proposition that a comet describes a parabola. I do not count the example of the attractive force exerted on something by "an endless row of marbles of such and such a kind" in Part IV, §8, and in the same section, the weight of a pillar composed of "as many slabs as their are cardinal numbers" as applications in physics, but rather as bits of pure mathematics.
[6] The logicists had other insights as well: they stressed the importance of the fact that one can count "abstract entities", e.g., numbers and equations, as well as emperors and cabbages, and also stressed that the things counted need not be adjacent in space or time or even exist in the lifetime of the person doing the counting; hence counting and forming sums of things counted are not physical operations. Wittgenstein's attacks on Russell's Principia in the Remarks are attacks on the significance Russell attached

to symbolic logic, not on these insights. I would also like to say that one thing that I like about The Remarks on the Foundations of Mathematics is that Wittgenstein treats empiricist views with a certain respect. It is true that, like Frege, he finds such views in the end completely inadequate, but he is willing to let us see the appeal of such views. He doesn't treat an empiricist view as simply a dumb view, as Frege does. On the contrary, empiricist views (and also finitist views) are treated as views to which one is naturally led by a desire for clarity, although Wittgenstein does agree with Frege that coming to see their inadquacy is an essential step to any further progress with the questions.

[7] Part IV, §25. I find this remark wrong, by the way, because it seems to me to forget something Wittgenstein himself elsewhere points out, which is that mathematatical propositions also have applications within mathematics. Wittgenstein himself seems to connect having an application with having a constructive proof here, but this is unjustified, unless one thinks of a very limited sort of application. Nevertheless, the fact that Wittgenstein made the remark shows that much more than the mastery of proof procedures is involved in the understanding of mathematical assertions, in his view.

[8] "On Wittgenstein's Philosophy of Mathematics", cited in n. 1.

[9] I realize that it is an anachronism to refer to Kripke in explaining one of Wittgenstein's insights, but the anachronism may help a present day reader.

[10] Kripke ref.

[11] Speaking for myself, I do not think that the mathematical notion of the infinite is always completely absent in the child's grasp that "you can always go on counting". Wittgenstein's discomfort with mathematical talk about what happens in infinite sequences may be influencing his perceptions here.

[12] See Philosophical Investigations §189, for the distinction between these two uses of "determine".

[13] See Michael Dummett's "Wittgenstein on Necessity: Some Reflections", in Reading Putnam, Peter Clark and Bob Hale, eds. (Oxford: Blackwells, 1994).

[14] Remarks on the Foundations of Mathematics, Part V, §34. It is important to contrast what Wittgenstein published on this question -- §516 of the Investigations -- with this unpublished paragraph in Remarks on the Foundations of Mathematics. Philosophical Investigations §516 reads "It seems clear that we understand the meaning of the question: 'Does the sequence 7777 occur in the development of □?' It is an English sentence; it can be shown what it means for 415 to occur in the development of □; and similar things. Well. our understanding of that question reaches just so far, one may say, as such explanations reach." Nothing here about what God can or cannot determine!

[15] The passage could, of course, be read in another way: there is a Last Judgement in time, and that is "the end of the world", and there is still more time available for calculation after the Last Judgement -- but then why should not human beings go on calculating in eternity?

[16] The fact that Wittgenstein is willing to consider the thought-experiment of imagining that humans go on and on calculating until "the end of the world" may also indicate that he is willing to allow us to idealize human abilities to calculate to the uttermost limits of physical possibility. (So that the relevant line here is not between what humans can calculate as opposed to what, say, Martians might be able to calculute, but between what it is within physical possibility to calculate and mere "mathematical possibility" of calculation. If so, this would go even more strongly against Dummett's reading.

[17] I have confined attention to rational values of t, r_1, and r_2 so as to avoid the objection that not all real numbers have names in the language. (A remark for the mathematically sophisticated.)

[18] At Gabriel Stolzenberg's suggestion, I emphasize that this is connected with the (possible) undecidability of the question whether the bodies will or will not collide.

[19] Marian Boykan Pour-El and Ian Richards, "The Wave Equation with Computable Initian Data Such that its Unique Solution is Not Computable," Advances in Mathematics 39 (1981), pp. 215-239.

[20] As remarked in an earlier note, computers have shown that 777 does occur in the decimal expansion of π. (So does 7777, Wittgenstein's example in PI §516.)

[21] Ian Hacking, Representing and Intervening [complete ref]

[22] [Give Hacking's "spraying" example]

[23] [say something about quantum mechanics]

[24] My own reasons for giving up the liberal form of verificationism that I defended in the early 1980s are given in "Pragmatism," Proceedings of the Aristotelian Society, vol. xcv, Part 3 (1995), 291-306.

[25] In a sense, all of Cora Diamond's essays in The Realistic Spirit (Cambridge, MA: MIT Press, 19XX) can be regarded as a non-verificationist interpretation of Wittgenstein. Recently she has tackled this issue explicitly, in "How Old Are These Bones?," [complete ref]

[26] See the references in n.1.

[27] Stanley Cavell, The Claim of Reason. [complete reference]

[28] Give ref to Yemima's paper

[29] Remarks on the Foundations of Mathematics, I, Appendix III, §8.

[30] Juliet Floyd and Hilary Putnam, "A Note on Wittgenstein's 'Notorious Paragraph' about the Gödel Theorem", The Journal of Philosophy, vol. xcvii, no. 11 (November 2000), pp. 624-632.

[31] At the time Wittgenstein wrote, it was not known that we can prove that 777 occurs in the decimal expansion of π.

[32] Remarks on the Foundations of Mathematics, Part V, §34. It is important to contrast what Wittgenstein published on this question -- §516 of the Investigations -- with this unpublished paragraph in Remarks on the Foundations of Mathematics. Philosophical Investigations §516 reads "It seems clear that we understand the meaning of the question: 'Does the sequence 7777 occur in the development of □?' It is an English sentence; it can be shown what it means for 415 to occur in the development of □; and similar things. Well. our understanding of that question reaches just so far, one may say, as such explanations reach." Nothing here about what God can or cannot determine!

[33] Earlier I pointed out that what certain philosophers have done is to take Wittgenstein to oppose not only metaphysical realism about rule following but also our commonsense realism about rule following. They have taken Wittgenstein to have doubts about the objectivity of rule-following when what Wittgenstein actually doubts is the need for and the very idea of an account of rule following over and above the ordinary account. (The ordinary account of rule following is, in my view, contained in Wittgenstein's description of how one teaches someone such concepts as "go on like this", "and so on", "and so on ad infinitum" at Philosophical Investigations §208. It is especially that Wittgenstein insists that when he teaches someone the concept of "going on like this" or "and so on", "I do not communicate less to him than I know myself". The attempt to explain these concepts – which function perfectly well in our lives – in terms of either platonic entities (such as the invisible rails of Philosophical Investigations §218) or in terms of mysterious mental powers is an attempt to explain something which we perfectly well understand in terms of something which Wittgenstein wants to "uncover" as "plain nonsense". The need for something mysterious to "support" or profound a "foundation for" our grammar is what is always under attack in Wittgenstein's writing, not the grammar itself.

[34] David Lewis,. Parts of Classes [complete ref]